Integral s parametrom

Naj bo $f(x,t)$: $[a,b]\times[c,d] \to \mathbb{R}$ funkcija:

Zveznost

$f(x,t)$ zvezna na $D_f \implies \mathrm{F}(x) = \int_c^d f(x,t)dt$ zvezna na [a,b].

Odvedljivost

$f(x,t)$ zvezna in zvezno parcialno odvedljiva po x na $D_f \implies \mathrm{F}(x) = \int_c^d f(x,t)dt$ odvedljiva na [a,b] in velja:

$$F'(x) = \int_c^d f_x(x,t)dt.$$

Integrabilnost

$f(x,t)$ zvezna na $D_f \implies \mathrm{F}(x) = \int_c^d f(x,t)dt$ integrabilna na [a,b] in velja:

$$\int_a^b F(x)dx = \int_a^b (\int_c^d f(x,t)dt)dx = \int_c^d (\int_a^b f(x,t)dx)dt.$$

Integral z variabilnimi mejami

Zveznost

Naj bo $f(x,t)$: $[a,b]\times[c,d] \to \mathbb{R}$ zvezna in u,v: $[a,b] \to [c,d]$ zvezni $\implies$ $F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$ zvezna na [a,b].

Odvedljivost

Naj bo $f(x,t)$: $[a,b]\times[c,d] \to \mathbb{R}$ zvezna in zvezno parcialno odvedljiva po x na D_f in naj bosta u,v: $[a,b] \to [c,d]$ odvedljivi $\implies$
$F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$ odvedljiva in velja:

$$F'(x) = \int_{u(x)}^{v(x)} f_x(x,t)dt + v'(x)f(x,v(x)) - u'(x)f(x,u(x)).$$

Izlimitirani integral s parametrom

Integral s parametrom $F(x) = \int_a^\infty f(x,t)dt$ je **enakomerno konvergenten** za x $\in$ [c,d], če za $\forall \varepsilon > 0\ \exists b > a$, da velja:

$$\left| \int_b^\infty f(x,t)dt \right| < \varepsilon \quad \forall x \in [c,d].$$

Weierstrass M-test

Če $\exists g : [a,\infty) \to \mathbb{R}$, da velja $|f(x,t)| < g(t)$ za $\forall x \in [c,d]$ in je $\int_a^\infty g(t)dt < \infty \implies F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergenten na [c,d]. V pomoč sta formuli:

$$\int_0^s t^{-\alpha}dt < \infty\ s > 0 \iff \alpha < 1.$$

$$\int_s^\infty t^{-\alpha}dt < \infty\ s > 0 \iff \alpha > 1.$$

Naj bo $f(x,t)$: $[c,d]\times[a,\infty) \to \mathbb{R}$ funkcija:

Zveznost

$f(x,t)$ zvezna na D_f in $F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergentna na [c,d] $\implies F$ zvezna na [c,d].

Odvedljivost

$f(x,t)$ zvezna in zvezno parcialno odvedljiva po x na D_f, $F(x) = \int_a^\infty f(x,t)dt$ konvergentna na [c,d] ter $F(x) = \int_a^\infty f_x(x,t)dt$ enakomerno konvergentna na [c,d] $\implies F'(x) = \int_a^\infty f_x(x,t)dt$.

Integrabilnost

$f(x,t)$ zvezna na D_f in $F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergentna na [c,d], potem velja:

$$\int_c^d F(x)dx = \int_c^d (\int_a^\infty f(x,t)dt)dx = \int_a^\infty (\int_c^d f(x,t)dx)dt.$$

Gamma in Beta funkciji

Gamma funkcija:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x \in (0,\infty)$$

1. $\Gamma(x+1) = x\Gamma(x)\ \forall x > 0$
2. $\Gamma(n) = (n-1)!\ \forall n \in \mathbb{N}$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Beta funkcija:

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \quad p,q > 0$$

1. $\beta(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}.\ \forall p,q > 0$
2. $\beta(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}du\ (t = \frac{u}{1+u})$
3. $\int_0^{\pi/2} sin^{2p-1}x \cdot cos^{2q-1}xdx = \frac{1}{2}\beta(p,q) \quad p,q > 0\ (t = \sin x^2)$
4. $\beta(1,q) = \frac{1}{q}$
5. $\beta(p+1,q) = \frac{p}{p+q}\beta(p,q)$
6. $\beta(p,q) = \beta(q,p)$

Eulerjeva refleksijska formula: $\beta(p,1-p) = \frac{\pi}{sin(\pi p)}\ p \in (0,1)$

$$\int_0^1 x^p(1-x)^q\ dx = \beta(p+1,q+1) = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$$
$$\int_0^\infty \frac{x^p}{(1+x)^q}\ dx = \beta(p+1,q-p-1) = \frac{\Gamma(p+1)\Gamma(q-p-1)}{\Gamma(q)}$$
$$\int_0^{\frac{\pi}{2}} \sin^p\varphi\cos^q\varphi\ d\varphi = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}$$

Večterni integral

Fubini

(1) Če f integrabilna na $[a,b]\times[c,d] \subset \mathbb{R}^2$ in $x \mapsto f(x,y)$ integrabilna na [a,b] za $\forall y \in$[c,d] in $y \mapsto f(x,y)$ integrabilna na [c,d] za $\forall x \in$ [a,b], potem:

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)dxdy = \int_a^b (\int_c^d f(x,y)dy)dx = \int_c^d (\int_a^b f(x,y)dx)dy$$

Analogno za $n \geq 3$.
(2) Naj bo $\Omega = \{(x,y,z) \in \mathbb{R}^3 | \ (x,y) \in D,\ g(x,y) \leq z \leq b(x,y)\}$, potem:

$$\iiint\limits_\Omega f(x,y,z)dxdydz = \iint\limits_D (\int_{g(x,y)}^{b(x,y)} f(x,y,z)dz)dxdy$$

(3) Fubinijev izrek v posplošenem integralu $\iiint\limits_\Omega f(x,y,z)dxdydz$ lahko uporabimo če:

- Ω omejeno območje in f omejena funkcija **ali**
- f pozitivna funkcija **ali**
- $\iiint\limits_\Omega |f(x,y,z)|\,dxdydz < \infty$

Nove spremenljivke

Jacobijeva matrika

Naj bo $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$, Jacobijevo matriko definiramo kot:

$$J_{f(x_1,\ldots,x_n)} = \begin{bmatrix} f_{1_{x_1}} & f_{1_{x_2}} & \ldots & f_{1_{x_n}} \\ f_{2_{x_1}} & f_{2_{x_2}} & \ldots & f_{2_{x_n}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ f_{m_{x_1}} & f_{m_{x_2}} & \ldots & f_{m_{x_n}} \end{bmatrix}$$

Vpeljava

Naj bosta $\Omega \subseteq \mathbb{R}^2$ in $\varphi : \Omega \to \mathbb{R}^2$ preslikava z zvezno odvedljivimi komponentami. Naj bo $detJ_\varphi \neq 0$ na Ω in naj bo $f : \varphi(\Omega) \to \mathbb{R}^2$ zvezna. Potem:

$$\iint\limits_{\varphi(\Omega)} f(x,y)dxdy = \iint\limits_\Omega f(\varphi(t,s)) \cdot |detJ_\varphi(t,s)|\,dtds$$

Polarne koordinate

$x = r\cos\varphi \quad y = r\sin\varphi \quad |detJ| = r \quad r \geq 0 \quad \varphi \in [0,2\pi]$

Cilindrične koordinate

$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z \quad |detJ| = r \quad r \geq 0 \quad \varphi \in [0,2\pi]$

Sferične koordinate

$x = R\cos\varphi\cos\vartheta \quad y = R\sin\varphi\cos\vartheta \quad z = R\sin\vartheta \quad R \geq 0 \quad \varphi \in [0,2\pi]$
kjer za ϑ velja:

- $\vartheta \in [-\frac{\pi}{2},\frac{\pi}{2}] \implies |detJ| = R^2\cos\vartheta$
- $\vartheta \in [0,\pi] \implies |detJ| = R^2\sin\vartheta$

Uporaba

$\rho(a)\ldots$ gostota v točki a, $D \subset \mathbb{R}^2$, $\Omega \subset \mathbb{R}^3$

Ploščina(S) ali volumen(V)

$S(D) = \iint\limits_D dxdy \quad V(\Omega) = \iiint\limits_\Omega dxdydz$

Masa(m)

$m(D) = \iint\limits_D \rho(x,y)dxdy \quad m(\Omega) = \iiint\limits_\Omega \rho(x,y,z)dxdydz$

Masno središče($\bar{a}$)

$\bar{x} = \frac{1}{m(D)}\iint\limits_D x\cdot\rho(x,y)dxdy \quad \bar{x} = \frac{1}{m(\Omega)}\iiint\limits_\Omega x\cdot\rho(x,y,z)dxdydz$
$\bar{y} = \frac{1}{m(D)}\iint\limits_D y\cdot\rho(x,y)dxdy \quad \bar{y} = \frac{1}{m(\Omega)}\iiint\limits_\Omega y\cdot\rho(x,y,z)dxdydz$
$\bar{z} = \frac{1}{m(\Omega)}\iiint\limits_\Omega z\cdot\rho(x,y,z)dxdydz$

Vztrajnostni moment (J)

okoli izhodišča: $J(D) = \iint\limits_D (x^2+y^2)\cdot\rho(x,y)dxdy$
okoli z-osi: $J_z(\Omega) = \iiint\limits_\Omega (x^2+y^2)\cdot\rho(x,y,z)dxdydz$
okoli y-osi: $J_y(\Omega) = \iiint\limits_\Omega (x^2+z^2)\cdot\rho(x,y,z)dxdydz$
okoli x-osi: $J_x(\Omega) = \iiint\limits_\Omega (y^2+z^2)\cdot\rho(x,y,z)dxdydz$

Konvergenca

$\iint_D f(x,y)dxdy$ je **absolutno konvergenten**,

če konvergira tudi $\iint_D |f(x,y)|\,dxdy$.

V primeru da $\iint_D f(x,y)dxdy$ konvergira, $\iint_D |f(x,y)|\,dxdy$ pa ne, pravimo da je integral **pogojno konvergenten**.

(1) $\iint_D |f(x,y)|\,dxdy$ konvergenten $\implies$ $\iint_D f(x,y)dxdy$ konvergenten.

(2) $\exists \iint_{\mathbb{R}^2} |f(x,y)|\,dxdy$ ali $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x,y)|\,dy$ ali $\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x,y)|\,dx$

potem $\exists \iint_{\mathbb{R}^2} f(x,y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)dx$.

Uporabne vrste

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad x \in \mathbb{R}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad x \in \mathbb{R}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad x \in \mathbb{R}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad x \in \mathbb{R}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n}x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 \cdots \qquad x \in (-1,1)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots \qquad x \in (-1,1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \qquad x \in (-1,1]$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} \cdots \qquad x \in [-1,1)$$

Opomba: $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-(n-1))}{n!}$

Integrali

Racionalne funkcije

$\int \frac{p(x)}{q(x)}dx, \qquad p(x),\ q(x)$ sta polinoma

1. Če je $st(q(x)) <= st(p(x))$ polinoma delimo
2. $q(x)$ razdelimo na linearne in kvadratne faktorje
3. Izraz pod integralom razcepimo na parcialne ulomke $\frac{p(x)}{q(x)} = \left[\frac{A_1}{x-a_1} + \cdots + \frac{A_{n_1}}{(x-a_1)^{n_1}}\right]$
4. Integriramo vsakega zase

$$k \geq 2 \qquad st(p(x)) \leq 2k-1$$

$$st(q(x)) \leq 2k-3 \qquad (ax^2+bx+c) \qquad \text{nerazcepen v } \mathbb{R}$$

$$I = \int \frac{p(x)}{(ax^2+bx+c)^k} = \int \frac{Ax+B}{ax^2+bx+c} + \frac{q(x)}{(ax^2+bx+c)^{k-1}}$$

A,B, $q(x)$ poiščemo tako da enačbo odvajamo.

Korenske funkcije

1. $\int f(\sqrt{ax+b})dx \qquad t = \sqrt{ax+b}$
2. $\int f(\sqrt{ax^2+bx+c})dx$

   (a) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ ga prevedemo na oblike:
   - Če je $a < 0 : \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$
   - Če je $a > 0 : \int \frac{dx}{\sqrt{x^2+c}} = \ln\left|x + \sqrt{x^2+c}\right|$

   (b) $\int \frac{p(x)}{\sqrt{ax^2+bx+c}} = q(x)\sqrt{ax^2+bx+c} + A\int \frac{dx}{\sqrt{ax^2+bx+c}}$
   $st(p(x)) - 1 = st(q(x))$ A, q(x) poiščemo z odvanjanjem
3. $\int \sqrt{a^2-x^2}dx \quad x = a\sin t \quad dx = a\cos t dt \quad t = \arcsin\frac{x}{a}$
4. $\int \sqrt{a^2+x^2}dx \quad x = a\,\mathrm{sh}\,t \quad dx = a\,\mathrm{ch}\,tdt \quad t = \mathrm{arsh}\,\frac{x}{a}$

Kotne funkcije

$$\int \sin(ax)\sin(bx)dx = \int -\frac{1}{2}\left[\cos(a+b)x - \cos(a-b)x\right]dx =$$

$$= -\frac{1}{2}\left[\frac{sin(a-b)x}{(a-b)} - \frac{sin(a+b)x}{(a+b)}\right]$$

$$\int \cos(ax)\cos(bx)dx = \int \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right]dx \dots$$

$$\int \sin(ax)\cos(bx)dx = \int \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right]dx \dots$$

Lihe in sode kotne funkcije $\int \cos^m x \sin^n x dx$

1. eno od števil m,n je liho (npr. $m = 2k+1$)

$$\int \cos^{2k} x \cos x \sin^n x dx = \int t^n(1-t^2)^k dt$$

$$t = \sin x \quad dt = \cos x dx$$

$$cos^{2k}x = (cos^2 x)^k = (1-t^2)^k$$

2. m, n sta oba soda, $m = 2m_1$,$n = 2n_1$

$$\int \cos^{2m_1} x \sin^{2n_1} x dx = \int (\cos^2 x)^{m_1}(\sin^2 x)^{n_1}dx =$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^{m_1}\left(\frac{1-\cos 2x}{2}\right)^{n_1} =$$

$$= \text{vsota integralov oblike} \int \cos^k 2x dx$$

kjer je $k \leq m_1 + n_1 = \frac{1}{2}(m+n) < m+1$
če je k lih gremo po 1 točki
če je k sod ponovimo postopek

3. $\int R(\cos x, \sin x)dx$ ($R\dots$ racinonalni izraz)

$$t = \tan\frac{x}{2} \qquad \cos x = \frac{1-t^2}{t^2+1}$$

$$\sin x = \frac{2t}{t^2+1} \qquad dx = \frac{2}{t^2+1}dt$$

$$t = \tan x \qquad \cos x = \frac{1}{\sqrt{t^2+1}}$$

$$\sin x = \frac{t}{\sqrt{t^2+1}} \qquad dx = \frac{dt}{t^2+1}$$

Tabela odvodov

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	a^x	$a^x \ln(a)$	e^x	e^x
$\frac{1}{x^n}$	$-\frac{n}{x^{n-1}}$	$\ln x$	$\frac{1}{x}$	$\log_a x$	$\frac{1}{xlna}$
$\csc x$	$-\cot(x)\csc(x)$	$\cos x$	$-\sin x$	$\tan x$	$\frac{1}{\cos^2 x}$
$\sec x$	$\tan(x)\sec(x)$	$\sin x$	$\cos x$	$\cot x$	$-\frac{1}{\sin^2 x}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Znane limite

$\lim_{x\to\infty} a^x = 0\ ,\ |a| < 1$ $\quad \lim_{x\to 0} x^x = 1$ $\quad \lim_{x\to\infty} \sqrt[x]{x} = 1$

$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a\ ,\ a > 0$ $\quad \lim_{x\to\infty} \frac{\ln x}{x} = 0$ $\quad \lim_{x\to 0} x \ln x = 0$

$\lim_{x\to\infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk}$ $\quad \lim_{x\to 0} \left(1 + kx\right)^{\frac{m}{x}} = e^{mk}$ $\quad \lim_{x\to 0} \frac{\sin x}{x} = 1$

nedoločene oblike
$\frac{0}{0}(L.H.), \frac{\infty}{\infty}(L.H.), 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

prevladujoči členi
$n^n \gg n! \gg q^n\ (|q| > 1) \gg n^a\ (a > 0) \gg ln(n)^a\ (a > 0)$

Tabela integralov

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}(n \neq -1)$	$\frac{1}{x}$	$\ln\|x\|$	e^x	e^x
$\sin x$	$-\cos x$	$\cos x$	$\sin x$	a^x	$\frac{a^x}{ln(a)}$
$\frac{1}{\cos^2 x}$	$\tan x$	$\frac{1}{\sin^2 x}$	$-\cot x$	$\cosh x$	$\sinh x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$	$\sinh^{-1} x$	$\frac{1}{1+x^2}$	$\arctan x$
$\sec x \tan x$	$\sec x$	$\csc x cot x$	$-\csc x$	$\tan x$	$\ln\|\sec x\|$
$\ln x$	$x \ln x - x$				

$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a} + C$ $\quad \int \frac{dx}{\sin^2 ax} = -\frac{\cot ax}{a} + C$

$\int \frac{dx}{a+x} = \ln|a + x| + C$ $\quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan(\frac{x}{a}) + C$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a}) + C$

$t = \tan(\frac{x}{2})$ $\quad \sin(x) = \frac{2t}{1+t^2}$ $\quad \cos(x) = \frac{1-t^2}{1+t^2}$ $\quad dx = \frac{2t}{1+t^2}$

$t = \tan(x)$ $\quad \sin(x) = \frac{t}{\sqrt{1+t^2}}$ $\quad \cos(x) = \frac{1}{\sqrt{1+t^2}}$ $\quad dx = \frac{t}{1+t^2}$

per partes $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$

racionalne funkcije $\int \frac{p(x)}{q(x)}dx$

če je $st(q(x)) <= st(p(x))$: (1) polinoma delimo, (2) $q(x)$ razdelimo na linearne in kvadratne faktorje, (3) izraz pod integralom razcepimo na parcialne ulomke, (4) integriramo vsakega zase

kotne funkcije $\int \cos^m x \sin^n x dx$

če je eno od števil m,n **liho**, uporabimo tisti člen za t substitucijo
če sta obe **sodi**, jih nadomestimo z identiteto polovičnih kotov

Izrazi

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

$a^n - b^n = (a - b)(a^{n-1} + \cdots + b^{n-1})$

$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$ $\quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$1 + a^{2n+1} = (1 + a)(1 - a + \cdots - a^{2n-1} + a^{2n})$

Potence, koreni, logaritmi

$a^n a^m = a^{n+m}$ $\quad a^n b^n = (ab)^n$ $\quad (a^n)^m = a^{nm}$ $\quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$

$\frac{a^n}{a^m} = a^{n-m}$ $\quad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$ $\quad a^{-n} = \frac{1}{a^n}$ $\quad ab^{-n} = \frac{a}{b^n}$

$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $\quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ $\quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ $\quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$

$(-a)^{2n} = a^{2n}$ $\quad \sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ $\quad (-a)^{2n+1} = -a^{2n+1}$

$\log_a x^n = nlog_a x$ $\quad log_b x = \frac{\log_a x}{log_a b}$ $\quad \log_a y = x \iff a^x = y$

$\log_a(xy) = \log_a(x) + \log_a(y)$ $\quad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

Kompleksna števila

$\alpha = a + bi$ $\quad \bar{\alpha} = a - bi$ $\quad \alpha\beta = (ac - bd) + (ad + bc)i$

$a = \frac{\alpha+\bar{\alpha}}{2}$ $\quad b = \frac{\alpha-\bar{\alpha}}{2i}$ $\quad \frac{\beta}{\alpha} = \frac{\beta\bar{\alpha}}{|\alpha|^2}$ $\quad \alpha\bar{\alpha} = |\alpha|^2$

$|\alpha| = \sqrt{a^2 + b^2}$ $\quad arg(\alpha) = atan2(a, b)$

$\alpha^n = |\alpha|^n e^{in\varphi}$ $\quad \alpha\beta = |\alpha|\,|\beta|\, e^{i(\varphi_{(\alpha)}+\varphi_{(\beta)})}$

$\alpha^n = |\alpha|^n \left(\cos(n\varphi) + i\sin(n\varphi)\right)$ $\quad \alpha^n = |\alpha|^n e^{i(n\varphi)}$

$\sqrt[n]{\alpha} = \sqrt[n]{|\alpha|}\left(\cos(\frac{\varphi}{n} + \frac{2\pi k}{n}) + i\sin(\frac{\varphi}{n} + \frac{2\pi k}{n})\right)$

$\sqrt[n]{\alpha} = \sqrt[n]{|\alpha|}e^{i(\frac{\varphi}{n}+\frac{2\pi k}{n})}$ $\quad {\scriptstyle k=0,1,2,\ldots,n-1}$

Kvadratna funkcija

$f(x) = ax^2 + bx + c$ $\quad f(x) = a(x - x_1)(x - x_2)$

$f(x) = a(x - p)^2 + q$ $\quad x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

teme T(p,q): $\quad p = -\frac{b}{2a}$ $\quad q = -\frac{b^2-4ac}{4a}$

Stožnice

parabola	$(y - q)^2 = \pm 2a(x - p)$	$d(T, \prod) =$	$\frac{\|ax+by+cz-d\|}{\sqrt{a^2+b^2+c^2}}$
krožnica	$(y - q)^2 = \pm 2a(x - p)$	$\vec{p}\cdot\vec{q} =$	$\|\vec{p}\|\,\|\vec{q}\|\cos\vartheta$
elipsa	$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$		
hiperbola	$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = \pm 1$		

Obsegi, površine, volumni

tip	obseg	površina	tip	površina	volumen
krog	$2\pi r$	πr^2	krogla	$4\pi r^2$	$\frac{4\pi r^3}{3}$
enak.trik.	$3a$	$\frac{a^2\sqrt{3}}{4}$	tetraeder	$a^2\sqrt{3}$	$\frac{a^3\sqrt{2}}{12}$
trapez	$a+b+c+d$	$\frac{a+c}{2}h$	valj	$2\pi r(r + h)$	$\pi r^2 h$
deltoid	$2a + 2b$	$ab\sin\alpha$	stožec	$2\pi r(r + s)$	$\frac{\pi r^2 h}{3}$

h = height, s = slant

Kotne funkcije

	$0°$	$30°$	$45°$	$60°$	$90°$					
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	Q1	Q2	Q3	Q4	S/L
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	+	+	-	-	L
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	+	-	-	+	S
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$	+	-	+	-	L
$\cot\alpha$	$-$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	+	-	+	-	L

$\sin\alpha = \frac{N}{H}$ $\quad \cos\alpha = \frac{P}{H}$ $\quad tan\alpha = \frac{N}{P}$ $\quad \cot\alpha = \frac{P}{N}$

$\sin^2\alpha = 1 - \cos^2\alpha$ $\quad \cos^2\alpha = 1 - \sin^2\alpha$

$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$ $\quad \frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1}{2}(1 - \cos\alpha)}$ $\quad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1}{2}(1 + \cos\alpha)}$ $\quad \cot\frac{\alpha}{2} = \frac{\sin\alpha}{1-\cos\alpha}$

$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$ $\quad 2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$ $\quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad \cos 2\alpha = 2\cos\alpha - 2\sin\alpha$

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$

$\cos\alpha \pm \cos\beta = 2\cos\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$

$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$

$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$

$\sin\alpha\sin\beta = -\frac{1}{2}(\cos(\alpha + \beta) - \cos(\alpha - \beta))$

$\sin\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

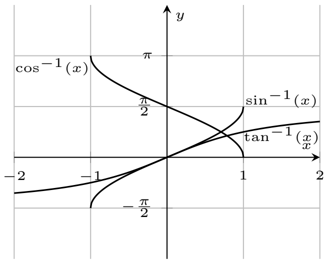
Hiperbolične funkcije

$\sinh x$	$\frac{e^x-e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} = \frac{1-e^{-2x}}{2e^{-x}}$	
$\cosh x$	$\frac{e^x+e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} = \frac{1+e^{-2x}}{2e^{-x}}$	
$\sinh^{-1} x$	$\ln(x + \sqrt{x^2 + 1})$	
$\cosh^{-1} x$	$\ln(x + \sqrt{x^2 - 1})$	$x \geq 1$
$\tanh^{-1} x$	$\frac{1}{2}\ln(\frac{1+x}{1-x})$	$\|x\| < 1$
$\coth^{-1} x$	$\frac{1}{2}\ln(\frac{x+1}{x-1})$	$\|x\| > 1$
$\cosh x + \sinh x = e^x$	$\cosh^2 x - \sinh^2 x = 1$	
$\cosh x - \sinh x = e^{-x}$		

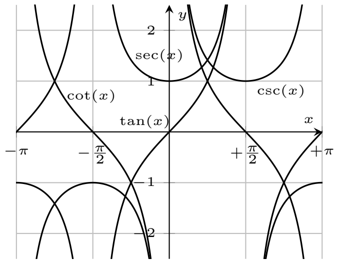
**ident. kotnih funkcij, vendar se pri sinh(x) * sinh(y) obrne predznak*

Krožne funkcije

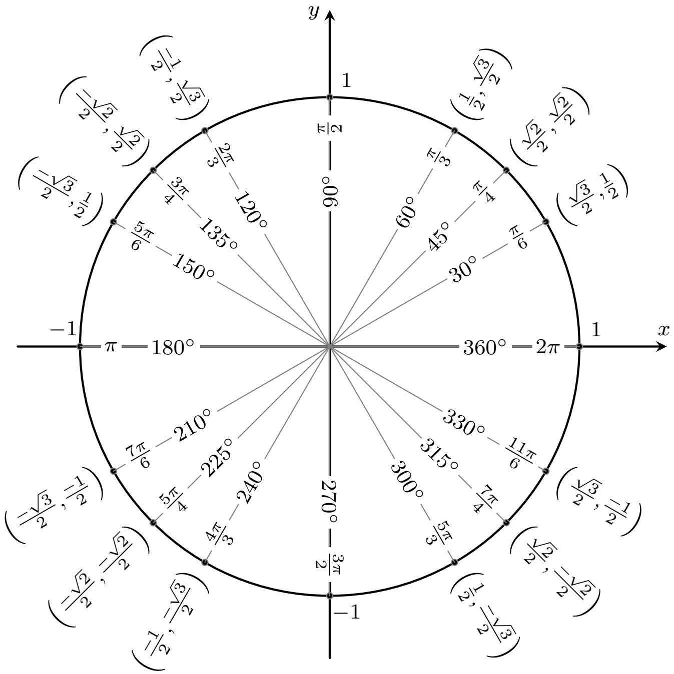
$\sin^{-1} x$	$D_f = [-1,1]$	$Z_f = [-\frac{\pi}{2},\frac{\pi}{2}]$
$\cos^{-1} x$	$D_f = [-1,1]$	$Z_f = [0,\pi]$
$\tan^{-1} x$	$D_f = (-\infty,\infty)$	$Z_f = (-\frac{\pi}{2},\frac{\pi}{2})$
$\cot^{-1} x$	$D_f = (-\infty,\infty)$	$Z_f = (0,\pi)$
$\sec^{-1} x$	$D_f = (-\infty,-1] \cup [1,\infty)$	$Z_f = [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$
$\csc^{-1} x$	$D_f = (-\infty,-1] \cup [1,\infty)$	$Z_f = [-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}]$



y
π
$\cos^{-1}(x)$
$\frac{\pi}{2}$
$\sin^{-1}(x)$
$\tan^{-1}(x)$
x
-2
-1
1
2
$-\frac{\pi}{2}$



y
2
$\sec(x)$
1
$\csc(x)$
$\cot(x)$
$\tan(x)$
x
$-\pi$
$-\frac{\pi}{2}$
$+\frac{\pi}{2}$
$+\pi$
-1
-2



y
x
1
-1
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60° $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{4}$ 45° $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{6}$ 30° $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{2\pi}{3}$ 120° $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{3\pi}{4}$ 135° $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{5\pi}{6}$ 150° $\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$
π 180°
360° 2π
$\frac{7\pi}{6}$ 210° $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$
$\frac{5\pi}{4}$ 225° $\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$
$\frac{4\pi}{3}$ 240° $\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$
270° $\frac{3\pi}{2}$
300° $\frac{5\pi}{3}$ $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$
315° $\frac{7\pi}{4}$ $\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$
330° $\frac{11\pi}{6}$ $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$